ON THE VOLUME OF ISOLATED SINGULARITIES

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Abstract. We give an equivalent definition of the local volume of an isolated singularity $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$ given in [BdFF12] in the $\mathbb{Q}$-Gorenstein case and we generalize it to the non-$\mathbb{Q}$-Gorenstein case. We prove that there is a positive lower bound depending only on the dimension for the non-zero local volume of an isolated singularity if X is Gorenstein. We also give a non-$\mathbb{Q}$-Gorenstein example with $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$, which does not allow a boundary Δ such that the pair (X,Δ) is log canonical.

1. Introduction

Let X be a normal variety and let $f : X \to X$ be a finite endomorphism, i.e. a finite surjective morphism of degree > 1. If X is projective, an abundant literature shows that the existence of an endomorphism imposes strong restrictions on the global geometry of X. In a recent paper [BdFF12], Boucksom, de Fernex and Favre introduce the volume $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$ of an isolated singularity and show that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ if there exists an endomorphism preserving the singularity. When X is $\mathbb{Q}$-Gorenstein, they show that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ if and only if X has log canonical singularities. For a better understanding of $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$, they propose two problems:

Problem A: Does there exist a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume?

Problem B: Is it true that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X,Δ) is log-canonical? (The converse being easily shown).

In this paper, we will give an alternative definition of the volume in the $\mathbb{Q}$-Gorenstein case via log canonical modification (the existence of these modifications is shown in [OX12]). By using the DCC for the volume established in [HMX13], we will give a positive answer to Problem A.

Theorem 1.1. *(See Theorem 3.3) There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

We will then generalize this new definition to the non-$\mathbb{Q}$-Gorenstein case. We will define the augmented volume $\mathrm{Vol}^+(X)$ as the liminf of the m-th limiting volumes $\mathrm{Vol}_m(X)$. We will show that the augmented volume $\mathrm{Vol}^+(X)$ is greater than or

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equal to the volume $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$. When there exists a boundary Δ on X such that the pair (X, Δ) is log canonical, we have the following theorem

Theorem 1.2. *(See Corollary 4.6) The following statements are equivalent:*

(1) *There exists a boundary Δ on X such that (X, Δ) is log canonical.*
(2) $\mathrm{Vol}_m(X) = 0$ *for some (hence any multiple of) integer* $m \geqslant 1$.

In Section 4.1, we will give a counterexample to Problem B.

Theorem 1.3. *(See Theorem 4.9) There exists a polarized smooth variety* (V, H) *such that the affine cone* $X = C(V, H)$ *has positive* $\mathrm{Vol}_m(X)$ *for any positive integer* m*, but* $\mathrm{Vol}_{BdFF}(X, 0) = 0$.

A new idea is needed to investigate whether $\mathrm{Vol}^+(X) = \mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$.

In [Fulger11], Fulger introduces a different invariant $\mathrm{Vol}_{\mathrm{F}}(X, 0)$ associated to an isolated singularity. It is shown that $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) \geqslant \mathrm{Vol}_{\mathrm{F}}(X, 0)$ with equality if X is $\mathbb{Q}$-Gorenstein. In [BdFF12, Example 5.4], an example is given where $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) > \mathrm{Vol}_{\mathrm{F}}(X, 0)$.

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2. Preliminaries

Throughout this paper, X is a normal variety over $\mathbb{C}$.

2.1. **Valuations of $\mathbb{Q}$-divisors.** Let X be a normal variety. A divisorial valuation v on X is a discrete valuation of the function field of X of the form $v = q\,\mathrm{val}_F$ where q is a positive integer and F is a prime divisor over X. Let $\mathcal{J} \subset \mathcal{K}$ be a finitely generated sub-$\mathcal{O}_X$-module of the constant sheaf of rational functions $\mathcal{K} = \mathcal{K}_X$ on X. For short, we will refer to $\mathcal{J}$ as a fractional ideal sheaf on X.

The valuation $v(\mathcal{J})$ of a non-zero fractional ideal sheaf $\mathcal{J} \subset \mathcal{K}$ along v is given by

$$v(\mathcal{J}) = \min\{v(\phi) | \phi \in \mathcal{J}(U), U \cap c_X(v) \neq \emptyset\}.$$

The valuation $v(I)$ of a formal linear combination $I = \sum a_k \cdot \mathcal{J}_k$ of fractional ideal sheaves $\mathcal{J}_k \subset \mathcal{K}$ along v is defined by $v(I) = \sum a_k \cdot v(\mathcal{J}_k)$, where a_k are real numbers.

The **$\natural$-valuation** (or **natural valuation**) along v of a Weil divisor D on X is $v^\natural(D) = v(\mathcal{O}_X(-D))$. If C is Cartier, then we have that $v^\natural(C) = v(C)$ and $v^\natural(C + D) = v(C) + v^\natural(D)$. Note also that, as $\mathcal{O}_X(D) \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$, we have that $v^\natural(D) + v^\natural(-D) \geqslant 0$.

To any non-trivial fractional ideal sheaf $\mathcal{J}$ on X, we associate the divisor $\mathrm{div}(\mathcal{J}) = \sum \mathrm{val}^\natural_E(\mathcal{J}) \cdot E$, where the sum is taken over all prime divisors E on X. Consider now a birational morphism $f : Y \to X$ from a normal variety Y. For any divisor D on X, the **$\natural$-pullback** (or **natural pullback**) of D to Y is given by $f^\natural D = \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)$. In the other words, $f^\natural D = \sum \mathrm{val}^\natural_E(D) \cdot E$, where the sum is taken over all prime divisors E on Y. In particular, $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.

For every divisor D on X and every positive integer m, it is shown in [dFH09, Lemma 2.8] that $m \cdot v^\natural(D) \geqslant v^\natural(mD)$ and

$$\inf_{k\geqslant 1} \frac{v^\natural(kD)}{k} = \liminf_{k\to\infty} \frac{v^\natural(kD)}{k} = \lim_{k\to\infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

Let D be a $\mathbb{Q}$-divisor on X. We define the **valuation along v of D** by

$$v(D) = \lim_{k\to\infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

If $f : Y \to X$ is a birational morphism from a normal variety Y, then the **pullback of D to Y** is defined by

$$f^*D = \sum \mathrm{val}_E(D) \cdot E,$$

where the sum is taken over all prime divisors E on Y. Notice that if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and m is a positive integer such that mD is Cartier, then

$$v(D) = \frac{v(mD)}{m} \quad \text{and} \quad f^*D = \frac{f^*(mD)}{m},$$

which coincide with the usual valuation and pullback of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If C is $\mathbb{Q}$-Cartier, then $v(C + D) = v(C) + v(D)$ and $f^*(C + D) = f^*C + f^*D$.

Lemma 2.1. *Let $f : Y \to X$ and $g : V \to Y$ be two birational morphisms of normal varieties. Then, for every divisor D on X, the divisor $(fg)^\natural D - g^\natural(f^\natural D)$ is effective and g-exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, then $(fg)^\natural D = g^\natural(f^\natural D)$. The similar statement applies to f^* and g^*.*

Proof. See Lemma 2.7 and Remark 2.13 in [dFH09]. □

2.2. **Relative canonical divisors.** We recall that a canonical divisor K_X on a normal variety X is, by definition, the (componentwise) closure of any canonical divisor of the regular locus of X. We also recall that X is said to be $\mathbb{Q}$-Gorenstein if some (equivalently, every) canonical divisor K_X is $\mathbb{Q}$-Cartier. For a proper birational morphism $f : Y \to X$ of normal varieties, we fix a canonical divisor K_Y on Y such that $f_*K_Y = K_X$. For any divisor D on X, we will write D_Y for the strict transform $f_*^{-1}D$ of D on Y.

For every $m \geqslant 1$, the **m-th limiting relative canonical $\mathbb{Q}$-divisor** $K_{m,Y/X}$ of Y over X is

$$K_{m,Y/X} = K_Y - \frac{1}{m} f^\natural(mK_X).$$

The **relative canonical $\mathbb{R}$-divisor** $K_{Y/X}$ of Y over X is

$$K_{Y/X} = K_Y - f^*K_X.$$

Clearly, $K_{Y/X}$ is the limsup of the $\mathbb{Q}$-divisors $K_{m,Y/X}$. A $\mathbb{Q}$-divisor Δ on X is said to be a **boundary**, if $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The **log relative canonical $\mathbb{Q}$-divisor** of (Y, Δ_Y) over (X, Δ) is given by

$$K^\Delta_{Y/X} = K_Y + \Delta_Y - f^*(K_X + \Delta).$$

Remark 2.2. Our definition of the relative canonical $\mathbb{R}$-divisor is different from the one in [dFH09]. In this paper, the relative canonical $\mathbb{R}$-divisor is defined as $K_{Y/X} = K_Y + f^*(-K_X)$. And $K_Y - f^*K_X$ is denoted by $K^-_{Y/X}$. It can be shown that, with this notation, $K_{Y/X} \geqslant K^-_{Y/X}$. But they are not equal in general. See [dFH09, Example 3.4].

For every integer $m \geqslant 1$, the **m-th limiting log discrepancy $\mathbb{Q}$-divisor** $A_{m,Y/X}$ of Y over X is

$$A_{m,Y/X} = K_Y + E_f - \frac{1}{m} f^{\natural}(mK_X),$$

where E_f is the reduced exceptional divisor of f. The **log discrepancy $\mathbb{R}$-divisor** $A_{Y/X}$ of Y over X is

$$A_{Y/X} = K_Y + E_f - f^* K_X.$$

The **log discrepancy $\mathbb{Q}$-divisor** of (Y, Δ_Y) over (X, Δ) is given by

$$A^{\Delta}_{Y/X} = K_Y + \Delta_Y + E_f - f^*(K_X + \Delta).$$

Consider a pair $(X, I = \sum a_k \cdot \mathcal{J}_k)$ where $\mathcal{J}_k$ are non-zero fractional ideal sheaves on X and a_k are real numbers. A **log resolution** of (X, I) is a proper birational morphism $f : Y \to X$ from a smooth variety Y such that for every k the sheaf $\mathcal{J}_k \cdot \mathcal{O}_Y$ is the invertible sheaf corresponding to a divisor E_k on Y, the exceptional locus $\mathrm{Ex}(f)$ of f is also a divisor, and $\mathrm{Ex}(f) \cup E$ has simple normal crossing, where $E = \bigcup \mathrm{Supp}(E_k)$. If Δ is a boundary on X, then a log resolution of the log pair $((X, \Delta), I)$ is given by a log resolution $f : Y \to X$ of (X, I) such that $\mathrm{Ex}(f) \cup E \cup \mathrm{Supp}(f^*(K_X + \Delta))$ has simple normal crossings. The log resolution always exists (see [dFH09, Theorem 4.2]).

Let X be a normal variety, and fix an integer $m \geqslant 2$. Given a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, a boundary Δ on X is said to be **m-compatible** for X with respect to f if:

(1) $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$,
(2) f is a log resolution for the log pair $((X, \Delta); \mathcal{O}_X(-mK_X))$, and
(3) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

Theorem 2.3. *For any normal variety X, any integer $m \geqslant 2$ and any log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, there exists an m-compatible boundary Δ for X with respect to f.*

Proof. See [dFH09, Theorem 5.4]. □

2.3. **Shokurov's b-divisors.** Let X be a normal variety. The set of all proper birational morphisms $\pi : X_\pi \to X$ modulo isomorphism is (partially) ordered by $\pi' \geqslant \pi$ if and only if π' factors through π, and any two proper birational morphisms can be dominated by a third one. The **Riemann-Zariski space** X is defined as the projective limit, $\mathcal{X} = \varprojlim_\pi X_\pi$. The group of **Weil b-divisors** over X is defined as $\mathrm{Div}(\mathcal{X}) = \varprojlim_\pi \mathrm{Div}(X_\pi)$, where $\mathrm{Div}(X_\pi)$ denotes the group of Weil divisors on X_π and the limit is taken with respect to the push-forwards. The group of **Cartier b-divisors** over X is defined as $\mathrm{CDiv}(\mathcal{X}) = \varinjlim_\pi \mathrm{CDiv}(X_\pi)$, where $\mathrm{CDiv}(X_\pi)$ denotes the group of Cartier divisors on X_π and the limit is taken with respect to the pull-backs. An element in $\mathrm{Div}_{\mathbb{R}}(\mathcal{X}) = \mathrm{Div}(\mathcal{X}) \otimes \mathbb{R}$ (resp. $\mathrm{CDiv}_{\mathbb{R}}(\mathcal{X}) = \mathrm{CDiv}(\mathcal{X}) \otimes \mathbb{R}$) will be called an $\mathbb{R}$-Weil b-divisor (resp. $\mathbb{R}$-Cartier b-divisor), and similarly with $\mathbb{Q}$ in place of $\mathbb{R}$. Clearly, a Weil b-divisor W over X consists of a family of Weil divisors $W_\pi \in \mathrm{Div}(X_\pi)$ that are compatible under push-forward. We say that W_π is the **trace** of W on the model X_π. Let C be an Cartier b-divisor. We say that $\pi : X_\pi \to X$ is a **determination** of C, if C can be obtained by pulling back C_π to models dominating π and pushing forward to other models, in which case we denote $C = \overline{C_\pi}$.

Let Z and W be two $\mathbb{R}$-Weil b-divisors over X. We say that $Z \leqslant W$, if for any model $\pi : X_\pi \to X$ we have $Z_\pi \leqslant W_\pi$. We say an $\mathbb{R}$-Cartier b-divisor is **relatively nef** over X, if its trace is relatively nef on one (hence any sufficiently high) model. An $\mathbb{R}$-Weil b-divisor W is nef over X if and only if there is a sequence of relatively nef $\mathbb{R}$-Cartier b-divisors over X such that the traces converge to the trace of W in the numerical class over X on each model.

Let $C_1, \dots, C_n$ be $\mathbb{R}$-Cartier b-divisors, where $n = \dim X$. Let f be a common determination. It is clear that the intersection number $C_{1,f} \cdot \ldots \cdot C_{n,f}$ is independent of the choice of f by projection formula. We define $C_1 \cdot \ldots \cdot C_n$ to be the above intersection number. If $W_1, \dots, W_n$ are relatively nef $\mathbb{R}$-Weil b-divisors over X, we define

$$W_1 \cdot \ldots \cdot W_n = \inf(C_1 \cdot \ldots \cdot C_n) \in [-\infty, \infty),$$

where the infimum is taken over all relatively nef $\mathbb{R}$-Cartier b-divisors C_i over X such that $C_i \geqslant W_i$ for each i.

Given a canonical divisor K_X on X, there is a unique canonical divisor K_{X_π} for each model $\pi : X_\pi \to X$ with the property that $\pi_* K_{X_\pi} = K_X$. Hence, a choice of K_X determines a **canonical b-divisor** $K_{\mathcal{X}}$ over X.

2.4. **Nef envelopes.** The **nef envelope** $\mathrm{Env}_X(D)$ **of an $\mathbb{Q}$-divisor** D on X is an $\mathbb{R}$-Weil b-divisor over X whose trace on a model $\pi : X_\pi \to X$ is $-\pi^*(-D)$. For a more general definition, see [BdFF12, Section 2]. If D is $\mathbb{Q}$-Cartier, then $\mathrm{Env}_X(D)$ is the $\mathbb{Q}$-Cartier b-divisor $\overline{D}$.

The **nef envelope** $\mathrm{Env}_{\mathcal{X}}(W)$ **of an $\mathbb{R}$-Weil b-divisor** W over X is the largest relatively nef $\mathbb{R}$-Weil b-divisor Z over X such that $Z \leqslant W$. It is well-defined by [BdFF12, Proposition 2.15]. It is clear that if $W_1 \leqslant W_2$, then $\mathrm{Env}_{\mathcal{X}}(W_1) \leqslant \mathrm{Env}_{\mathcal{X}}(W_2)$.

The **log discrepancy b-divisor** is defined as

$$A_{\mathcal{X}/X} = K_{\mathcal{X}} + E_{\mathcal{X}/X} + \mathrm{Env}_X(-K_X),$$

where the trace of $E_{\mathcal{X}/X}$ in any model π is equal to the reduced exceptional divisor E_π over X. It is clear that the trace of $A_{\mathcal{X}/X}$ on a model $\pi : X_\pi \to X$ is $A_{X_\pi/X}$. Similarly, for every integer $m \geqslant 1$, we define the **m-th limiting log discrepancy b-divisor** $A_{m,\mathcal{X}/X}$ to be a $\mathbb{Q}$-Weil b-divisor whose trace on a model $\pi : X_\pi \to X$ is $A_{m,X_\pi/X}$. It is easy to check that $A_{m,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}$ and $A_{\mathcal{X}/X}$ is the limsup of $A_{m,\mathcal{X}/X}$.

The **volume of the singularity** on X is defined by

$$\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = -\,\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})^n.$$

It is shown in [BdFF12] that if X has isolated singularity, then $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$ is a well-defined non-negative finite real number.

2.5. **Log canonical modification.** Suppose (X, Δ) is a pair such that X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A birational projective morphism $f : Y \to X$ is called a log canonical modification of (X, Δ) if

(1) $(Y, \Delta_Y + E_f)$ is log canonical,
(2) $K_Y + \Delta_Y + E_f$ is f-ample,

where Δ_Y is the strict transform of Δ and E_f is the reduced exceptional divisor of f. It is shown in [OX12] that the log canonical modification exists uniquely up to

isomorphism for any log pair (X, Δ). Clearly, if $f' : Y' \to X$ is a log resolution of the pair (X, Δ), then

$$Y \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_* \mathcal{O}_{Y'}(m(K_{Y'} + \Delta_{Y'} + E_{f'})).$$

Lemma 2.4. *Let (X, Δ) be a pair as above. Let $f : Y \to X$ be the log canonical model. Write $f^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Y + \Delta_Y + B$, and $B = \sum b_i B_i$ as the sum of distinct prime divisors. We let $B^{>1}$ be the nonzero divisor $\sum_{b_i > 1} b_i B_i$, then $\mathrm{Supp}(B^{>1}) = \mathrm{Ex}(f)$. In particular, $\mathrm{Ex}(f) \subset Y$ is of pure codimension 1.*

Proof. See [OX12, Lemma 2.4]. □

3. $\mathbb{Q}$-Gorenstein Case

Assume that X is a $\mathbb{Q}$-Gorenstein normal variety with isolated singularities. We pick $\Delta = 0$ and suppose that $f : Y \to X$ is the log canonical modification of X. By the Negativity Lemma (see [KM98, Lemma 3.39]), $F = f^*K_X - K_Y - E_f \geqslant 0$. We define

$$\mathrm{Vol}(X) = F \cdot (K_Y + E_f)^{n-1} = -(K_Y + E_f - f^*K_X)^n \geqslant 0,$$

since $K_Y + E_f$ is f-ample and F is f-exceptional.

Remark 3.1. This definition can be extended to the case that X has isolated non-log-canonical locus.

Theorem 3.2. *If X is a $\mathbb{Q}$-Gorenstein normal variety who has isolated singularities, then $\mathrm{Vol}_{BdFF}(X, 0) = \mathrm{Vol}(X)$.*

Proof. We show that $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor and equals to $\overline{A_{Y/X}}$ where $f : Y \to X$ is the log canonical modification of $(X, 0)$. Then the theorem follows immediately.

We only need to show that on a high enough model $f' : Y' \to X$ which factors through $f : Y \to X$ via $g : Y' \to Y$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_{f'}$ equals to $g^*A_{Y/X}$. By passing to a higher model, we may assume that Y' is smooth, hence D is $\mathbb{R}$-Cartier. By the definition, $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is relatively nef over X, then D is the limit of a sequence of f'-nef $\mathbb{R}$-Cartier divisors. Thus, D is f'-nef.

First, we show that $g^*A_{Y/X} \leqslant D$. Since (Y, E_f) is log canonical, we have that

$$\begin{aligned} & A_{Y'/X} - g^*A_{Y/X} \\ = \; & K_{Y'} + E_{f'} - f'^*K_X - g^*(K_Y + E_f - f^*K_X) \\ = \; & K_{Y'} + (E_f)_{Y'} + E_g - g^*(K_Y + E_f) \\ \geqslant \; & 0, \end{aligned}$$

where $(E_f)_{Y'}$ is the strict transform of E_f on Y'. As $A_{Y/X}$ is f-ample, we have that $g^*A_{Y/X}$ is f'-nef. We can conclude that $\overline{A_{Y/X}}$ is a nef $\mathbb{Q}$-Cartier b-divisor which is less than or equal to $A_{\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A_{Y/X}} \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. In particular, $g^*A_{Y/X} \leqslant D$.

Now, $g^*A_{Y/X} \leqslant D \leqslant A_{Y'/X}$. We have that $A_{Y'/X} - g^*A_{Y/X} = K_{Y'} + (E_f)_{Y'} + E_g - g^*(K_Y + E_f)$ is g-exceptional, hence so is $D - g^*A_{Y/X}$. Since D is f'-nef, we have that $D - g^*A_{Y/X}$ is g-nef. By the Negativity Lemma, $D - g^*A_{Y/X} \leqslant 0$. Therefore, $D = g^*A_{Y/X}$. □

Theorem 3.3. *There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

Proof. Suppose X is a Gorenstein normal variety with isolated singularities and $f : Y \to X$ is its log canonical modification. Let $F = f^*K_X - K_Y - E_f = \sum a_i \cdot E_i$, where E_i are f-exceptional divisors. Since X is Gorenstein, we have that a_i are positive integers by Lemma 2.4. If $\mathrm{Vol}(X) > 0$, then $F \neq 0$, hence $F \geqslant E_f$. We have that

$$\mathrm{Vol}(X) \geqslant E_f \cdot (K_Y + E_f)^{n-1} = ((K_Y + E_f)|_{E_f})^{n-1} = (K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}.$$

Since (Y, E_f) is log canonical, by [Kollár92, 16.6], the coefficients of $\mathrm{Diff}_{E_f}(0)$ lies in $\{0, 1\} \cup \{1 - \frac{1}{m} | m \geqslant 2\}$, which is a DCC set. By [HMX13, Proposition 5.1], we have that $(K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}$ lies in a DCC set. The theorem follows. □

4. Non-$\mathbb{Q}$-Gorenstein Case

Let X be a normal variety which has only isolated singularities. For any integer $m \geqslant 2$, fix a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$. By Theorem 2.3, we can find a boundary Δ such that $K^{\Delta}_{Y/X} = K_{m,Y/X}$. Let $f_{lc} : Y_{lc} \to X$ be the log canonical modification of the pair (X, Δ). Then

$$Y_{lc} \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_*\mathcal{O}_Y(m(K_Y + \Delta_Y + E_f)).$$

Assuming that Δ' is another m-compatible boundary for X with respect to f and $f'_{lc} : Y'_{lc} \to X$ is the corresponding log canonical modification, we have that $K^{\Delta}_{Y/X} = K^{\Delta'}_{Y/X}$. Hence, $\Delta_Y - \Delta'_Y = f^*(\Delta - \Delta')$. Now,

$$\begin{aligned} & f_*\mathcal{O}_Y(m(K_Y + \Delta'_Y)) \\ = \ & f_*\mathcal{O}_Y(m(K_Y + \Delta_Y - f^*(\Delta - \Delta'))) \\ = \ & f_*\mathcal{O}_Y(m(K_Y + \Delta_Y)) \otimes \mathcal{O}_X(m(\Delta' - \Delta)) \end{aligned}$$

for sufficiently divisible m, as $\Delta - \Delta'$ is $\mathbb{Q}$-Cartier. Thus, there is a natural X-isomorphism $\sigma : Y_{lc} \to Y'_{lc}$ such that $f_{lc} = f'_{lc} \circ \sigma$. Fix a common resolution of Y and Y_{lc}, $\tilde{f} : \tilde{Y} \to X$, as in the following diagram:

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y_{lc} \arrow[dl, "f_{lc}"] \\ & X & \end{tikzcd}$$

Noticing that $\tilde{Y}$ is also a common resolution of Y and Y'_{lc}, we have that the morphism $s : \tilde{Y} \to Y$ is independent of the choice of Δ.

Theorem 4.1. *The $\mathbb{R}$-Weil b-divisor $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor. If Δ is m-compatible for X with respect to $\tilde{f} : \tilde{Y} \to X$, then*

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) = \overline{A^{\Delta}_{Y_{lc}/X}}.$$

Proof. We will mimic the proof of Theorem 3.2. We only need to show that on a high enough model $\rho : Z \to X$ which factors through $\tilde{f} : \tilde{Y} \to X$ by $\pi : Z \to \tilde{Y}$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_\rho$ equals to $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$. By passing to a higher model, we may assume that Z is smooth, hence D is $\mathbb{R}$-Cartier. Since $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ is relatively nef over X, we have that D is the limit of a sequence of ρ-nef $\mathbb{R}$-Cartier divisors. Thus, D is ρ-nef.

First, we show that $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant D$. Since Δ is m-compatible for X with respect to $\tilde{f}$, we have that

$$K_{\tilde{Y}} + \Delta_{\tilde{Y}} - \tilde{f}^*(K_X + \Delta) = K_{\tilde{Y}} - \frac{1}{m}\tilde{f}^{\natural}(mK_X).$$

Hence,

$$-\frac{1}{m}\rho^{\natural}(mK_X) = \pi^*\left(-\frac{1}{m}\tilde{f}^{\natural}(mK_X)\right) = \pi^*\Delta_{\tilde{Y}} - \rho^*(K_X+\Delta),$$

by Lemma 2.1 and the fact that $\tilde{Y}$ is smooth. Now, the difference

$$\begin{aligned}
& A_{m,Z/X} - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \\
= \;& K_Z + E_\rho - \frac{1}{m}\rho^{\natural}(mK_X) - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) + \rho^*(K_X+\Delta) \\
= \;& K_Z + E_\rho + \pi^*\Delta_{\tilde{Y}} - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\
= \;& (K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_{t\circ\pi} - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})) + (\pi^*\Delta_{\tilde{Y}} - \Delta_Z)
\end{aligned}$$

Since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical, we have that the first term is effective and $(t\circ\pi)$-exceptional. As Δ_Z is the strict transform of the effective divisor $\Delta_{\tilde{Y}}$, the second term is effective and π-exceptional, hence, $(t\circ\pi)$-exceptional. We conclude that $A_{m,Z/X} - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is effective and $(t\circ\pi)$-exceptional. Since $A^{\Delta}_{Y_{lc}/X}$ is f-ample, we have that $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is ρ-nef. We can conclude that $\overline{A^{\Delta}_{Y_{lc}/X}}$ is a nef $\mathbb{Q}$-Cartier b-divisor which is less than or equal to $A_{m,\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A^{\Delta}_{Y_{lc}/X}} \leqslant \mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$. In particular, $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant D$.

Now, $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant D \leqslant A_{m,Z/X}$. Recall that $A_{m,Z/X} - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is $(t\circ\pi)$-exceptional, hence so is $D - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$. Since D is ρ-nef, we have that $D - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is $(t\circ\pi)$-nef. By the Negativity Lemma, $D - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant 0$. Therefore, $D = (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$. □

We can define the volume of singularities of X as follow:

Definition 4.2. The m-th limiting volume of singularity of X is

$$\mathrm{Vol}_m(X) = -\,\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})^n.$$

Corollary 4.3. *In the setting of Theorem 4.1, if Δ is m-compatible for X with respect to $\tilde{f}$, then*

$$\mathrm{Vol}_m(X) = -(A^{\Delta}_{Y_{lc}/X})^n = -A^{\Delta}_{Y_{lc}/X} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} \geqslant 0.$$

Proof. The first equation is straightforward by Theorem 4.1 and the definition of intersection number. The second equation is valid since $A^{\Delta}_{Y_{lc}/X}$ is f_{lc}-exceptional. By the Negativity Lemma, we have that $A^{\Delta}_{Y_{lc}/X} \leqslant 0$. Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, we have the inequality in the corollary. □

For an arbitrary boundary Δ on X, we have the following inequalities.

Proposition 4.4. *Suppose that Δ is a boundary on X, m is the index of $K_X + \Delta$ and $f : Y \to X$ is the log canonical modification of (X, Δ). Then*

(1) $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$,
(2) $\mathrm{Vol}_m(X) \leqslant -(A^{\Delta}_{Y/X})^n$.

Proof. For any model $\pi : X_\pi \to X$, we have that

$$\pi^*(m(K_X + \Delta)) + \pi^\natural(-m\Delta) = \pi^\natural(mK_X).$$

Hence,

$$\begin{aligned} A_{m,X_\pi/X} &= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^\natural(mK_X) \\ &= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^\natural(-m\Delta) - \frac{1}{m}\pi^*(m(K_X + \Delta)) \\ &\geqslant K_{X_\pi} + E_\pi + \Delta_{X_\pi} - \pi^*(K_X + \Delta) = A^{\Delta}_{X_\pi/X}. \end{aligned}$$

Thus, as b-divisors, $A_{m,\mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X}$, hence,

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}).$$

On the other hand, for any model $f' : Y' \to Y$ factoring through f via $g : Y' \to Y$, we have that $A^{\Delta}_{Y'/X} \geqslant g^* A^{\Delta}_{Y/X}$, since $(Y, \Delta_Y + E_f)$ is log canonical. As b-divisors, $A^{\Delta}_{\mathcal{X}/X} \geqslant \overline{A^{\Delta}_{Y/X}}$. As $K_Y + \Delta_Y + E_f$ is f-ample, we have that $\overline{A^{\Delta}_{Y/X}}$ is relatively nef over X. Thus, $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$. We proved (1).

Since both $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ and $\overline{A^{\Delta}_{Y/X}}$ are relatively nef and exceptional over X, (2) follows from the inequality between intersection numbers. □

Remark 4.5. In the last proposition, one can show that $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) = \overline{A^{\Delta}_{Y/X}}$.

Corollary 4.6. *The following statements are equivalent:*

(1) *There exists a boundary Δ on X such that (X, Δ) is log canonical.*
(2) $\mathrm{Vol}_m(X) = 0$ *for some (hence any multiple of) integer $m \geqslant 1$.*

Proof. (1) $\Rightarrow$ (2). Suppose $m(K_X + \Delta)$ is Cartier. By Proposition 4.4,

$$A_{m,\mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X} \geqslant 0,$$

since (X, Δ) is log canonical. As 0 is a relatively nef b-divisor over X, we have that $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant 0$. On the other hand, by Theorem 4.1 and the Negativity Lemma, $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \leqslant 0$. Hence, $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) = 0$. We can conclude that $\mathrm{Vol}_m(X) = 0$.

(2) $\Rightarrow$ (1). Let Δ be an m-compatible boundary for X with respect to $\tilde{f}$ in the setting of Theorem 4.1. By Theorem 2.3, such a boundary always exists. Since $\mathrm{Vol}_m(X) = 0$, by Corollary 4.3, we have that

$$-A^{\Delta}_{Y_{lc}/X} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} = 0.$$

Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, this is equivalent to $A^{\Delta}_{Y_{lc}/X} = 0$. Thus, we have that

$$f^*_{lc}(K_X + \Delta) = K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}.$$

For any model $\rho : Z \to X$ factoring through f_{lc} via $\pi : Z \to Y_{lc}$, we have that

$$\begin{aligned} A^{\Delta}_{Z/X} &= K_Z + \Delta_Z + E_\rho - \rho^*(K_X + \Delta) \\ &= K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_\pi - \pi^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\ &\geqslant 0, \end{aligned}$$

since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical. Therefore, (X, Δ) is log canonical. □

For any two positive integers m and l and any model $f : Y \to X$, since

$$\frac{1}{m} f^\natural(mK_X) \geqslant \frac{1}{lm} f^\natural(lmK_X) \geqslant f^*K_X,$$

we have that

$$A_{m,Y/X} \leqslant A_{lm,Y/X} \leqslant A_{Y/X},$$

hence

$$A_{m,\mathcal{X}/X} \leqslant A_{lm,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}.$$

By the definition of nef envelope, we have that

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{lm,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}).$$

Since they are both exceptional over X by Theorem 4.1, we have the following inequality of volumes:

$$\mathrm{Vol}_m(X) \geqslant \mathrm{Vol}_{lm}(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X, 0).$$

Definition 4.7. The augmented volume of singularities on X is

$$\mathrm{Vol}^+(X) = \liminf_m \mathrm{Vol}_m(X) = \lim_{k\to\infty} \mathrm{Vol}_{k!}(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X, 0).$$

Remark 4.8. While it is proved in the appendix of [BdFF12] that the intersection number is continuous, it is not clear that $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ converge to $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. It is interesting to have an example with $\mathrm{Vol}^+(X) > \mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$.

4.1. **Cone singularities.** We will give a counterexample to Problem B in this section.

Let (V, H) be a non-singular projective polarized variety of dimension $n-1$. The vertex 0 is the isolated singularity of the normal variety

$$X = \mathrm{Spec} \bigoplus_{m\geqslant 0} H^0(V, \mathcal{O}_V(mH)).$$

Blowing up 0 gives a resolution of singularities for X that we denote by Y. The induced map $f : Y \to X$ is isomorphic to the contraction of the zero section E of the total space of the vector bundle $\mathcal{O}_V(H)$. Let $\pi : Y \to V$ be the bundle map. We have that $E \cong V$. The co-normal bundle of E in Y is

$$\mathcal{O}_E(-E) \cong \mathcal{O}_V(H).$$

Let Γ be a boundary on X, Γ_Y be the strict transform of Γ on Y and $\Delta = \Gamma_Y|_E$. Since $A^\Gamma_{Y/X}$ is exceptional, we may assume that $A^\Gamma_{Y/X} = -aE$ for some rational number a. Restricting to E, we have that

$$K_V + \Delta \sim_{\mathbb{Q}} aH,$$

by the adjunction formula. On the other hand, assuming that Δ is an effective $\mathbb{Q}$-Cartier divisor on V such that $\Delta \sim_{\mathbb{Q}} -K_V + aH$, we may set $\Gamma = C_\Delta$ and get that $A^{C_\Delta}_{Y/X} = -aE$.

Let C be an elliptic curve, U be a semi-stable vector bundle on C of rank 2 and degree 0 and $V = \mathbb{P}(U)$ be the ruled surface over C. The nef cone $\mathrm{Nef}(V)$ and pseudo-effective cone $\overline{\mathrm{NE}}(V)$ are the same. They are spanned by the section C_0 corresponding to the tautological bundle $\mathcal{O}_{\mathbb{P}(U)}(1)$ and a fiber F of the ruling (for details, see e.g. [Lazarsfeld04, Section 1.5.A]). Moreover, as in [Shokurov00, Example 1.1], if C' is an effective curve on V such that $C' \equiv mC_0$ for some positive integer m, then $C' = mC_0$.

Theorem 4.9. *Let V be the ruled surface as above. Fix an ample divisor H on V. Let X be the affine cone over (V, H). Then $\mathrm{Vol}^+(X) = 0$, hence $\mathrm{Vol}_{BdFF}(X, 0) = 0$. But there is no effective $\mathbb{Q}$-divisor Γ such that (X, Γ) is log canonical.*

Proof. Fix an ample divisor H on V. Since $K_V \sim -2C_0$, we have that $-K_V + aH$ is ample for any rational number $a > 0$. Let D be a smooth curve in $|n(-K_V + aH)|$ for some sufficiently large positive integer n, and set $\Delta = \frac{1}{n}D$. Then $(Y, \pi^*\Delta + E)$ is the log canonical modification of (X, C_Δ), since $(K_Y + \pi^*\Delta + E)|_E \sim_{\mathbb{Q}} aH$ is ample. Suppose m is the index of $K_X + C_\Delta$. By Proposition 4.4 (2),

$$\mathrm{Vol}_m(X) \leqslant -(A^{C_\Delta}_{Y/X})^3 = (aE)^3 = a^3H^2.$$

As $a \to 0$, we conclude that $\mathrm{Vol}^+(X) = 0$, hence $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$.

If (X, Γ) is log canonical for some effective $\mathbb{Q}$-divisor Γ on X, then $A^{\Gamma}_{Y/X} = -aE \geqslant 0$, hence $a \leqslant 0$. On the other hand, let $\Delta = \Gamma_Y|_E \in \overline{\mathrm{NE}}(V)$. Then

$$\Delta \sim_{\mathbb{Q}} -K_V + aH \sim_{\mathbb{Q}} 2C_0 + aH.$$

Since $\Delta \geqslant 0$, we have $a \geqslant 0$ and hence $a = 0$. Thus $A^{\Gamma}_{Y/X} = 0$, and $(Y, \Gamma_Y + E)$ is log canonical. But $\Delta = \Gamma_Y|_E$ is an effective $\mathbb{Q}$-divisor linearly equivalent to $2C_0$, and hence $\Delta = 2C_0$, a contradiction. □

In [dFH09, Definition 7.1], a normal variety X is defined to be log canonical if for one (hence any sufficiently divisible) positive integer m, the m-th limiting log discrepancy b-divisor $A_{m,\mathcal{X}/X} \geqslant 0$. And in [ibid, Proposition 7.2], they proved that X is log canonical if and only if there is a boundary Δ such that the pair (X, Δ) is log canonical. It is natural to ask whether this definition is equivalent to the one requiring that the log discrepancy b-divisor $A_{\mathcal{X}/X} \geqslant 0$.

Corollary 4.10. *Let X be the affine cone over (V, H) as in Theorem 4.9. Then $A_{\mathcal{X}/X} \geqslant 0$, but X is not log canonical.*

Proof. The corollary follows immediately from the fact that $A_{\mathcal{X}/X} \geqslant 0$ is equivalent to $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ (see [BdFF12, Proposition 4.19] or Corollary 4.6). □

4.2. **Approach without boundary.** It is conjectured that for a normal variety X which has only isolated singularities, there is a log canonical modification $f : Y \to X$ in the sense that $K_Y + E_f$ is f-ample and (Y, E_f) is log canonical. In [BH12, Proposition 2.4], it is proved that $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ if and only if f is an isomorphism in codimension 1.

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